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CUET · MATHEMATICS · CLASS XI · CODE 319

Limits and Derivatives

CUET unit: Limits and Derivatives

By **UniDrill** · NCERT-grounded study material

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📌 Snapshot

- Calculus begins here: the derivative is defined through limits, starting from the falling-body example $s = 4.9t^2$ and the search for instantaneous velocity (NCERT §12.1–§12.2, p. 217–220).
- It builds the apparatus: left/right hand limits, algebra of limits, limits of polynomial and rational functions, the standard limit $(x^n - a^n)/(x - a) \to n \cdot a^{(n-1)}$, and the two trig limits $\sin x/x \to 1$ and $(1 - \cos x)/x \to 0$.
- It then defines the derivative from first principles as $f'(a) = \lim_{h\to0} [f(a+h) - f(a)]/h$, gives its geometric meaning as the slope of the tangent, and develops the algebra of derivatives (sum, difference, product (Leibnitz), quotient) together with derivatives of x^n, sin x, cos x, tan x and other polynomial/trigonometric forms.
- CUET tests this unit heavily because almost every calculus problem in higher classes builds on its standard limits, first-principle technique, and product/quotient rules.

📖 Detailed Notes

2.1 Core concepts

- Calculus studies how the value of a function changes as the points in its domain change. The core ideas covered are: the intuitive notion of a derivative, the (naive) limit, the algebra of limits, the formal derivative, and the standard derivative formulae (NCERT §12.1, p. 217).
- The free-fall law $s = 4.9t^2$ with successive average velocities computed over shrinking intervals ending at and starting from $t = 2$ yields a common limiting value, called the instantaneous velocity at $t = 2$; this is the prototype of a derivative (NCERT §12.2, Tables 12.1–12.3, p. 218–219).
- Geometrically, the sequence of chords C_iB_i / AC_i approaches the slope of the tangent at A as the time intervals shrink to zero, so the instantaneous velocity equals the slope of the tangent to $s = 4.9t^2$ at $t = 2$ (NCERT §12.2, Fig 12.1, p. 219–220).
- For a function f, $\lim_{x\to a} f(x) = l$ means f(x) approaches l as x approaches a; x may approach a from the left or from the right, giving left-hand limit $\lim_{x\to a^-} f(x)$ and right-hand limit $\lim_{x\to a^+} f(x)$. The limit exists at $x = a$ if and only if these two one-sided limits coincide (NCERT §12.3, p. 220–221).

- The example $f(x) = 1$ for $x \leqslant 0$ and $f(x) = 2$ for $x > 0$ shows that the LHL (= 1) and RHL (= 2) at 0 can differ, in which case $\lim_{x\to0} f(x)$ does not exist even though $f(0)$ is defined (NCERT §12.3, Fig 12.3, p. 221).
- In general the value of the function at a point and its limit at that point need not be equal: e.g. $f(x) = x + 2$ for $x \neq 1$, $f(1) = 0$ gives $\lim_{x\to1} f(x) = 3 \neq f(1)$ (NCERT §12.3 Illustration 10, Fig 12.7, p. 227).
- **Algebra of limits (Theorem 1):** If $\lim f(x)$ and $\lim g(x)$ both exist at a, then $\lim[f \pm g] = \lim f \pm \lim g$; $\lim[f\cdot g] = \lim f \cdot \lim g$; $\lim[f/g] = \lim f / \lim g$ provided $\lim g \neq 0$; and $\lim(\lambda\cdot f) = \lambda\cdot\lim f$ for any constant λ (NCERT §12.3.1, p. 228).
- **Polynomials:** Using $\lim_{x\to a} x = a$ and induction, $\lim_{x\to a} x^n = a^n$; therefore for any polynomial f, $\lim_{x\to a} f(x) = f(a)$ (NCERT §12.3.2, p. 228–229).
- **Rational functions:** For $f(x) = g(x)/h(x)$ with $h(a) \neq 0$, $\lim_{x\to a} f(x) = g(a)/h(a)$. If $h(a) = 0$ and $g(a) \neq 0$, the limit does not exist. If both vanish, cancel common $(x - a)$ factors and re-evaluate; the limit is 0 if the order of vanishing of g exceeds that of h, undefined if it is less (NCERT §12.3.2, p. 229–230).
- **Standard algebraic limit (Theorem 2):** For any positive integer n, $\lim_{x\to a} (x^n - a^n)/(x - a) = n\cdot a^{(n-1)}$. The result extends to any rational exponent n with $a > 0$ (NCERT §12.3.2 Remark, p. 232–233).
- **Sandwich Theorem (Theorem 4):** If $f(x) \leqslant g(x) \leqslant h(x)$ on a common domain and $\lim_{x\to a} f = \lim_{x\to a} h = l$, then $\lim_{x\to a} g = l$ (NCERT §12.4, p. 234).
- The geometric inequality $\cos x < \sin x / x < 1$ for $0 < x < \pi/2$ is established from the circle figure (area of triangle OAC < area of sector OAC < area of triangle OAB) (NCERT §12.4, Fig 12.10, p. 235).
- **Standard trigonometric limits (Theorem 5):** $\lim_{x\to0} \sin x / x = 1$ and $\lim_{x\to0} (1 - \cos x)/x = 0$. The first follows from the sandwich theorem; the second uses the identity $1 - \cos x = 2 \sin^2(x/2)$ (NCERT §12.4, p. 235–236).
- **Derivative at a point (Definition 1):** $f'(a) = \lim_{h\to0} [f(a+h) - f(a)]/h$, provided the limit exists; $f'(a)$ measures the rate of change of f at a (NCERT §12.5, p. 240).
- **Derivative as a function (Definition 2 — first principle):** $f'(x) = \lim_{h\to0} [f(x+h) - f(x)]/h$ wherever the limit exists; also denoted $d/dx\,(f(x))$, dy/dx, or $D(f(x))$ (NCERT §12.5, p. 242).
- **Geometric meaning:** $f'(a)$ equals $\tan\psi$, the slope of the tangent to $y = f(x)$ at $(a, f(a))$; the chord PQ tends to this tangent as $h \to 0$ (NCERT §12.5, Fig 12.11, p. 241–242).
- **Algebra of derivatives (Theorem 5 of §12.5.1):** $(u \pm v)' = u' \pm v'$; product (Leibnitz) rule $(uv)' = u'v + uv'$; quotient rule $(u/v)' = (u'v - uv')/v^2$, valid where $v \neq 0$ (NCERT §12.5.1, p. 244).
- **Power rule (Theorem 6):** $d/dx\,(x^n) = n\,x^{(n-1)}$ for any positive integer n (proved via the binomial expansion of $(x + h)^n$); the remark extends it to all real n (NCERT §12.5.1, p. 245–246).

- **Polynomial derivative (Theorem 7):** $d/dx\,(a_n x^n + \cdots + a_1 x + a_0) = n\,a_n\,x^{n-1} + (n-1)\,a_{n-1}\,x^{n-2} + \cdots + a_1$ (NCERT §12.5.2, p. 246).
- **Standard trig derivatives:** $d/dx\,(\sin x) = \cos x$ (Example 16, p. 247); $d/dx\,(\cos x) = -\sin x$ (Summary, p. 255); $d/dx\,(\tan x) = \sec^2 x$ (Example 17, p. 247–248); $d/dx\,(\cot x) = -\operatorname{cosec}^2 x$ (Example 21, p. 251–252).

2.2 Definitions to memorise

Term	Definition	Page
Left-hand limit	$\lim_{x\to a^-} f(x)$ — expected value of f at a from values to the left of a	221
Right-hand limit	$\lim_{x\to a^+} f(x)$ — expected value of f at a from values to the right of a	221
Limit at a point	Common value of LHL and RHL if they coincide; denoted $\lim_{x\to a} f(x)$	221
Polynomial function	$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$ with $a_n \neq 0$, n a natural number	228
Rational function	$f(x) = g(x)/h(x)$ with g, h polynomials and $h(x) \neq 0$	229
Sandwich Theorem	If $f \leqslant g \leqslant h$ and $\lim f = \lim h = l$, then $\lim g = l$	234
Derivative at a (Defn 1)	$f'(a) = \lim_{h\to 0} [f(a+h) - f(a)]/h$, if this limit exists	240
First principle (Defn 2)	$f'(x) = \lim_{h\to 0} [f(x+h) - f(x)]/h$, wherever the limit exists	242
Product (Leibnitz) rule	$(uv)' = u'v + uv'$	244
Quotient rule	$(u/v)' = (u'v - uv')/v^2,\ v \neq 0$	244–245
Power rule	$d/dx\,(x^n) = n\,x^{n-1}$	245

Standard limits to memorise (Summary, p. 254–255):

Limit	Value
$\lim_{x\to a} (x^n - a^n)/(x - a)$	$n\cdot a^{n-1}$
$\lim_{x\to 0} \sin x / x$	1
$\lim_{x\to 0} (1 - \cos x)/x$	0

Standard derivatives (Summary, p. 255): $d/dx\,(x^n) = n\,x^{n-1}$; $d/dx\,(\sin x) = \cos x$; $d/dx\,(\cos x) = -\sin x$.

2.3 Diagrams / processes to remember

- **Fig 12.1 (p. 219):** Plot of $s = 4.9t^2$ with shrinking time intervals $h_1, h_2, \cdots$; the chord slopes C_iB_i/AC_i tend to the slope of the tangent at A — the geometric origin of the derivative.
- **Fig 12.2 (p. 220):** Graph of $h(x) = (x^2 - 4)/(x - 2)$, an undefined point at $x = 2$ with limit 4 — shows that limit can exist where the function value does not.
- **Fig 12.3 (p. 221):** Step-like function with $f(x) = 1$ for $x \leqslant 0$ and 2 for $x > 0$; visual demonstration that LHL ≠ RHL ⇒ limit does not exist.
- **Fig 12.6 (p. 227):** Piecewise function with jump at 0 — LHL = −2, RHL = +2, so limit does not exist though $f(0) = 0$.
- **Fig 12.7 (p. 227):** $f(x) = x + 2$ for $x \neq 1$, $f(1) = 0$ — limit at 1 is 3, value is 0; limit ≠ value.
- **Fig 12.10 (p. 235):** Unit circle with sector OAC; the area inequality $(1/2)OA \cdot CD < (1/2)\pi(OA)^2(x/\pi) < (1/2)OA \cdot AB$ delivers $\cos x < \sin x / x < 1$.
- **Fig 12.11 (p. 241):** Secant PQ through (a, f(a)) and (a+h, f(a+h)); as $h \to 0$, secant tends to tangent at P, so slope $(QR/PR) \to f'(a)$.

2.5 Key formulas & theorems

Formula	Statement	NCERT page
Limit existence	LHL = RHL = L	221
Polynomial limit	lim_{x→a} p(x) = p(a)	228
Rational limit (denom ≠ 0)	g(a)/h(a)	229
Algebra lim(sum)	lim(f) + lim(g)	228
Algebra lim(product)	lim(f) · lim(g)	228
Algebra lim(quotient)	lim(f)/lim(g) if lim(g) ≠ 0	228
$(x^n - a^n)/(x - a)$	n a^(n−1)	232
sin x / x	1 as $x \to 0$	235
(1 − cos x)/x	0 as $x \to 0$	235
tan x / x	1 as $x \to 0$	237
Sandwich theorem	$f \leqslant g \leqslant h$, lim f = lim h = L ⇒ lim g = L	234
Derivative at a	lim_{h→0} (f(a+h) − f(a))/h	240
First principle	f′(x) = lim_{h→0} (f(x+h) − f(x))/h	242
Sum rule	(u + v)′ = u′ + v′	244
Product (Leibnitz) rule	(uv)′ = u′v + uv′	244
Quotient rule	$(u/v)' = (u'v - uv')/v^2$	244

Formula	Statement	NCERT page
Power rule	d/dx (x^n) = n x^(n−1)	245
Polynomial derivative	term by term	246
d/dx (sin x)	cos x	247
d/dx (cos x)	−sin x	255
d/dx (tan x)	$\sec^2 x$	247
d/dx (cot x)	$-\text{cosec}^2 x$	251
Constant rule	d/dx (c) = 0	247
Slope of tangent	f′(a) at (a, f(a))	242
Linearity of derivative	(cf)′ = cf′	244

2.6 Solved examples (NCERT-grounded)

Example A (NCERT Example 2(ii), p. 230). Evaluate lim_{x→2} ($x^3 - 4x^2 + 4x$)/($x^2 - 4$).

Step 1 — factor: numerator = $x(x - 2)^2$; denominator = (x − 2)(x + 2). Step 2 — cancel: limit = lim_{x→2} x(x − 2)/(x + 2). Step 3 — substitute: 2·0/4 = **0**.

Example B (NCERT Example 3(i), p. 233). Evaluate lim_{x→1} ($x^{15} - 1$)/($x^{10} - 1$).

Step 1 — rewrite as ratio of standard limits: [($x^{15} - 1$)/(x − 1)] ÷ [($x^{10} - 1$)/(x − 1)]. Step 2 — apply Theorem 2: $15 \cdot 1^{14}$ / $10 \cdot 1^9$ = 15/10. Step 3 — simplify: = **3/2**.

Example C (NCERT Example 4(i), p. 236). Evaluate lim_{x→0} sin 4x / sin 2x.

Step 1 — rewrite: [(sin 4x)/(4x)] · [(2x)/(sin 2x)] · 2. Step 2 — limits: each ratio → 1. Step 3 — product: 1·1·2 = **2**.

Example D (NCERT Example 5, p. 240). First-principle derivative of f(x) = 3x at x = 2.

Step 1 — set up: f′(2) = lim_{h→0} [3(2+h) − 3·2]/h. Step 2 — simplify numerator: 3h. Step 3 — limit: 3h/h = 3 ⇒ **f′(2) = 3**.

Example E (NCERT Example 15, p. 247). Differentiate f(x) = (x + 1)/x.

Step 1 — quotient rule: u = x + 1, v = x, u′ = 1, v′ = 1. Step 2 — apply formula: (1·x − (x+1)·1)/x^2. Step 3 — simplify: (x − x − 1)/x^2 = **$-1/x^2$**.

2.4 Common confusions / NTA trap points

- A function may be defined at x = a while lim_{x→a} f(x) does not exist (e.g. Fig 12.3, Fig 12.6) — students wrongly assume lim = f(a) always (NCERT §12.3 and Summary point 3, p. 221, 254).
- Conversely, the limit may exist where the function is not defined, e.g. ($x^2 - 4$)/(x − 2) at x = 2 (NCERT §12.3 illustration, p. 220).

- For a rational function with 0/0 form, you must factor and cancel before substituting; substituting blindly gives an undefined expression even though the limit may be a finite number or zero (NCERT §12.3.2, Example 2(ii)–(iv), p. 230–231).
- Theorem 2 lim $(x^n - a^n)/(x - a) = n \cdot a^{(n-1)}$ often hides under substitution: students forget to put it in the form f(y) where y → 1 (e.g. $((1+x)^{(1/2)} - 1)/x$ via y = 1 + x, NCERT Example 3(ii), p. 233).
- The sin x / x and (1 – cos x)/x limits are about angles in **radians** only; using degrees is a classic trap (NCERT §12.4 proof, p. 234–236).
- Product rule (uv)′ = u′v + uv′ is **not** u′v′ — a frequent NTA distractor (NCERT §12.5.1, p. 244).
- Quotient rule has the sign and order fixed: numerator is u′v – uv′, not uv′ – u′v; reversing the order changes the sign (NCERT §12.5.1, p. 244–245).
- d/dx (cos x) = – sin x carries a minus sign that students drop (NCERT Summary, p. 255).
- Confusing the derivative as a number (f′(a)) with the derivative as a function (f′(x)); the first depends on a specific point, the second is itself a function of x.
- Forgetting to take limit before substitution in indeterminate forms 0/0; direct substitution gives "undefined", not the limit value.
- Misreading "instantaneous velocity" as average velocity over a finite interval; the limit gives the instantaneous (point-in-time) velocity.
- Mistakenly using $(x^n - a^n)/(x - a) \to n \cdot a^n$ instead of $n \cdot a^{(n-1)}$; the exponent drops by one when the derivative is taken.

🎯 Practice MCQs

📊 PYQ Alignment

CUET (UG) Mathematics has carried this chapter as a heavyweight unit since 2022, with roughly 10–12 MCQs in a 50-question section coming from limits-and-derivatives directly or as building blocks. Recurring formats include direct evaluation of 0/0 rational limits using factoring or Theorem 2, computation of sin(kx)/sin(mx) and tan-based trig limits, first-principle derivatives of simple polynomials and trigonometric functions, and product/quotient-rule differentiation of expressions of the type (polynomial × trig) or (linear/linear).