FREE EDITION · NOTES + 3 SAMPLE MCQS

CUET · MATHEMATICS · CLASS XII · CODE 319

Application of Integrals

CUET unit: Application of Integrals

By **UniDrill** · NCERT-grounded study material

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📌 Snapshot

- Definite integrals compute the area enclosed by a curve, the coordinate axes and given ordinates/abscissae — a problem elementary geometry formulae cannot handle for curved boundaries.
- It develops the vertical-strip formula $A = \int_a^b y\,dx = \int_a^b f(x)\,dx$ and the horizontal-strip formula $A = \int_c^d x\,dy = \int_c^d g(y)\,dy$ from the "thin strip" intuition.
- Standard worked results — area of circle $x^2 + y^2 = a^2$ is πa^2 and area of ellipse $x^2/a^2 + y^2/b^2 = 1$ is πab — are derived using symmetry plus a first-quadrant integral multiplied by 4.
- Sign handling is emphasised: if the curve dips below the x-axis, the raw integral is negative and the area is its absolute value; when a curve crosses the x-axis on $[a,b]$, the total area is $|A_1| + A_2$.
- CUET regularly tests the basic area formula, the circle/ellipse standard results, area between a line and the x-axis, and the convention of treating signed areas as absolute values.

📖 Detailed Notes

2.1 Core concepts

- Elementary geometry gives areas of triangles, rectangles, trapezia and circles but is inadequate for regions enclosed by general curves; integral calculus fills this gap (NCERT §8.1, p. 292). The fundamental theorem of calculus, established in the previous chapter, converts area computations into antiderivative evaluations.
- The definite-integral-as-limit-of-a-sum (from the Integrals chapter) gives areas under simple curves, areas between arcs of standard circles, parabolas and ellipses, and areas bounded by such curves with lines (NCERT §8.1, p. 292).
- To find the area bounded by $y = f(x)$, the x-axis and ordinates $x = a$, $x = b$, take a thin vertical strip of height y and width dx; the elementary area is $dA = y\,dx$ (NCERT §8.2, p. 292, Fig 8.1). The strip approximates the region locally and the integral is the limit as $dx \to 0$.
- Adding these elementary areas across the region PQRSP gives the master formula $A = \int_a^b dA = \int_a^b y\,dx = \int_a^b f(x)\,dx$ (NCERT §8.2, p. 293).

- When it is more convenient, take a horizontal strip of width dy; the area of the region bounded by $x = g(y)$, the y-axis and the lines $y = c$, $y = d$ is $A = \int_c^d x\, dy = \int_c^d g(y)\, dy$ (NCERT §8.2, p. 293, Fig 8.2). Switching to horizontal strips is useful when the curve is naturally written as x in terms of y (e.g., $y^2 = 4ax$ becomes $x = y^2/(4a)$).
- If the curve lies below the x-axis between $x = a$ and $x = b$ (i.e. $f(x) < 0$), the value of $\int_a^b f(x)\, dx$ is negative; the area is taken as its absolute value $|\int_a^b f(x)\, dx|$ (NCERT §8.2 Remark, p. 293, Fig 8.3).
- If some portion of the curve is above the x-axis and some below (Fig 8.4) with $A_1 < 0$ and $A_2 > 0$, the area bounded by the curve, x-axis and the ordinates $x = a$, $x = b$ is $A = |A_1| + A_2$ — the two pieces are computed separately and then added in absolute value (NCERT §8.2, p. 293–294).
- Example 1 — area enclosed by the circle $x^2 + y^2 = a^2$: by symmetry about both axes, total area $= 4 \int_0^a y\, dx = 4 \int_0^a \sqrt{a^2 - x^2}\, dx$, since $y = \sqrt{a^2 - x^2}$ in the first quadrant; evaluating using $\int\sqrt{a^2 - x^2}\, dx = (x/2)\sqrt{a^2 - x^2} + (a^2/2) \sin^{-1}(x/a)$ gives πa^2 (NCERT §8.2 Example 1, p. 294).
- The same area πa^2 is obtained alternatively by taking horizontal strips: $4 \int_0^a x\, dy = 4 \int_0^a \sqrt{a^2 - y^2}\, dy = \pi a^2$ (NCERT §8.2 Example 1 (Alternatively), p. 295, Fig 8.6).
- Example 2 — area of the ellipse $x^2/a^2 + y^2/b^2 = 1$: by symmetry the area is 4 × (first-quadrant area bounded by curve, x-axis and ordinates $x = 0$, $x = a$); using $y = (b/a)\sqrt{a^2 - x^2}$, area $= 4 \int_0^a (b/a)\sqrt{a^2 - x^2}\, dx = (4b/a)\cdot(a^2/2)\cdot(\pi/2) = \pi ab$ (NCERT §8.2 Example 2, p. 295).
- Horizontal-strip alternative for the ellipse gives the same answer πab via $4 \int_0^b x\, dy = 4\ (a/b) \int_0^b \sqrt{b^2 - y^2}\, dy = \pi ab$ (NCERT §8.2 Example 2 Alternatively, p. 296, Fig 8.8).
- Miscellaneous Example 3 — area bounded by $y = 3x + 2$, x-axis and ordinates $x = -1$, $x = 1$: the line meets the x-axis at $x = -2/3$, lying below the x-axis on $(-1, -2/3)$ and above on $(-2/3, 1)$; required area = $|\int_{-1}^{-2/3} (3x+2)\, dx| + \int_{-2/3}^{1} (3x+2)\, dx = 1/6 + 25/6 = 13/3$ (NCERT §8 Miscellaneous Example 3, p. 297, Fig 8.9).
- Miscellaneous Example 4 — area bounded by $y = \cos x$ between $x = 0$ and $x = 2\pi$: cosine is positive on $[0, \pi/2]$ and $[3\pi/2, 2\pi]$ and negative on $[\pi/2, 3\pi/2]$, so area = $\int_0^{\pi/2} \cos x\, dx + |\int_{\pi/2}^{3\pi/2} \cos x\, dx| + \int_{3\pi/2}^{2\pi} \cos x\, dx = 1 + 2 + 1 = 4$ (NCERT §8 Miscellaneous Example 4, p. 297, Fig 8.10).
- Summary statement: area bounded by $y = f(x)$, x-axis and lines $x = a$, $x = b$ (with $b > a$) is $\int_a^b y\, dx = \int_a^b f(x)\, dx$; area bounded by $x = \phi(y)$, y-axis and lines $y = c$, $y = d$ is $\int_c^d x\, dy = \int_c^d \phi(y)\, dy$ (NCERT §8 Summary, p. 298).

- Geometric intuition is essential: always sketch the region first, identify which strip type (vertical or horizontal) is simplest, locate intersections with the x-axis if sign-changes occur, then integrate piecewise with absolute values.
- Areas between two curves (implicit, made explicit in exercises): if $f(x) \geqslant g(x)$ on $[a, b]$, then area between $y = f(x)$ and $y = g(x)$ is $\int_a^b (f(x) - g(x))\,dx$. The "top minus bottom" mnemonic captures this.
- For curves intersecting at more than two points, partition the integration interval at every intersection and apply (top – bottom) on each subinterval. This avoids sign errors when one curve crosses the other.
- The applications extend in higher mathematics to volumes of revolution (washer and shell methods), arc length, and surface area — but Class XII NCERT focuses purely on planar area.
- The "limit of a sum" derivation of area is mathematically equivalent to the definite integral by the Fundamental Theorem of Calculus, ensuring that integration and area are two faces of the same coin.
- A worked tip for the standard integrals: $\int\sqrt{(a^2 - x^2)}\,dx$ and $\int\sqrt{(a^2 + x^2)}\,dx$ are reduction formulas with $\sin^{-1}$ and ln (or $\sinh^{-1}$) primitives; memorise the former because it appears in every circle/ellipse derivation.
- A geometric pitfall: when the curve does not cross the x-axis on $[a, b]$ but lies entirely below, the integral is negative; report area as its absolute value. When it crosses, partition the interval; sign-handling errors here are the single biggest source of wrong answers on CUET.
- Symmetry shortcut: for even functions on $[-a, a]$, area = $2 \int_0^a f(x)\,dx$; for odd functions, the signed integral is zero but the area is still $2 \int_0^a |f(x)|\,dx$.
- For a parabola $y = ax^2$ between $x = -p$ and $x = p$: by symmetry, area = $2 \int_0^p ax^2\,dx = (2/3)\,a\,p^3$; the same parabola "opens upward" so area lies above x-axis when $a > 0$.
- Always confirm units of measure (e.g., square units) match the geometric setting; numerical answers should be positive real numbers.

2.2 Definitions to memorise

Term	Definition	Page
Elementary area	$dA = y\,dx$ (vertical strip)	292
Area under curve (vertical)	$\int_a^b f(x)\,dx$	293
Area under curve (horizontal)	$\int_c^d g(y)\,dy$	293
Below-axis area		$\int_a^b f(x)\,dx$
Crossing-axis area		A_1
Circle area	πa^2	294
Ellipse area	πab	295

Term	Definition	Page
Quarter circle	$\pi a^2/4$	294
Standard integral $\sqrt{(a^2-x^2)}$	$(x/2)\sqrt{(a^2-x^2)} + (a^2/2)\sin^{-1}(x/a)$	294
Vertical strip	Perpendicular to x-axis	292
Horizontal strip	Perpendicular to y-axis	293
Signed area	Integral value with sign	293
Geometric area	Non-negative quantity	293
First-quadrant area	Used with symmetry, ×4 for full	294
Bounding ordinates	$x = a$, $x = b$	293
Bounding abscissae	$y = c$, $y = d$	293
Sketch first	Visual prerequisite for area problems	297
Crossing point	x where $f(x) = 0$	297
Parabola area ($y^2 = 4ax$)	$\int$ horizontal strips $x = y^2/(4a)$	296
Below-zero integrand	Multiplied by −1 before integration	293
Symmetric region	Reduce work via 2× or 4×	294
Standard area triangle	(1/2)·base·height	297
Vertical-strip width	dx	292
Horizontal-strip width	dy	293
Strip area dA	y dx or x dy	292

2.3 Diagrams / processes to remember

- **Fig 8.1 (p. 292):** region PQRSP under $y = f(x)$ between $x = a$ and $x = b$, with an arbitrary vertical strip of height y and width dx illustrating $dA = y\,dx$.
- **Fig 8.2 (p. 293):** region bounded by $x = g(y)$, y-axis and lines $y = c$, $y = d$, with a horizontal strip of width dy illustrating $dA = x\,dy$.
- **Fig 8.3 (p. 293):** curve $y = f(x)$ lying entirely below the x-axis on $[a, b]$, motivating the absolute-value convention for area.
- **Fig 8.4 (p. 294):** curve with one portion above and one below the x-axis, illustrating $A = |A_1| + A_2$.
- **Fig 8.5 (p. 294):** circle $x^2 + y^2 = a^2$ with the first-quadrant strip used to derive πa^2 by $4 \int_0^a \sqrt{(a^2 - x^2)}\,dx$.
- **Fig 8.6 (p. 295):** same circle handled with horizontal strips ($4 \int_0^a \sqrt{(a^2 - y^2)}\,dy$) showing the alternative derivation.
- **Fig 8.7 (p. 295):** ellipse $x^2/a^2 + y^2/b^2 = 1$ with first-quadrant vertical strip used to derive πab.

- **Fig 8.8 (p. 296):** ellipse handled with horizontal strips for the alternative derivation of πab.
- **Fig 8.9 (p. 297):** line $y = 3x + 2$ cutting the x-axis at $x = -2/3$, splitting the region between $x = -1$ and $x = 1$ into a below-axis triangle ACBA and an above-axis triangle ADEA.
- **Fig 8.10 (p. 297):** graph of $y = \cos x$ on $[0, 2\pi]$ partitioned at $x = \pi/2$ and $x = 3\pi/2$, used to compute the total bounded area as $1 + 2 + 1 = 4$.
- **Process — area under a curve:** (i) sketch the region; (ii) decide strip orientation (vertical or horizontal); (iii) find intersection of curve with x-axis (if applicable); (iv) split the integral at sign changes; (v) integrate each piece with |·| as needed; (vi) sum.
- **Process — area of a closed region:** identify symmetries; compute area in one symmetric portion; multiply by the symmetry factor.

2.4 Common confusions / NTA trap points

- Forgetting that $\int_a^b f(x)\,dx$ is **negative** when the curve lies below the x-axis and that the area must be reported as the **absolute value** (NCERT §8.2 Remark, p. 293).
- When a curve crosses the x-axis on $[a, b]$, integrating in one stroke gives $A_1 + A_2$ (with cancellation) instead of the correct total area $|A_1| + A_2$ (p. 294, Fig 8.4) — Example 3 and Example 4 are textbook traps for this.
- Mixing up the two formulae: $\int y\,dx$ (vertical strips, area against the x-axis) versus $\int x\,dy$ (horizontal strips, area against the y-axis). The choice depends on which axis the strip is perpendicular to (p. 293, Fig 8.2).
- Standard-area confusion: students sometimes write area of an ellipse as $\pi a^2 b^2$, $\pi(a + b)^2/4$, or πr^2 with $r = (a + b)/2$. The correct area is πab (Example 2, p. 295).
- In the circle derivation, dropping the factor of 4 (which comes from the symmetry about both axes — only the first-quadrant area is computed by $\int_0^a \sqrt{(a^2 - x^2)}\,dx$) is a frequent slip (Example 1, p. 294).
- Forgetting the absolute value when integrating a function with sign change. The integral $\int$ from 0 to 2π of cos x is 0; the area is 4.
- Confusing "between two curves" with "under a curve". Area between y = f(x) and y = g(x) where $f \geqslant g$ on [a, b] is $\int(f - g)\,dx$, not $\int f\,dx - \int g\,dx$ (though algebraically the two are equal).
- Wrong direction of integration: $\int_b^a f\,dx = -\int_a^b f\,dx$. For area, b should be the upper limit.
- Mis-applying symmetry: the circle is symmetric about both axes (4-fold); the ellipse also; the parabola $y^2 = 4ax$ is symmetric about x-axis (2-fold); odd functions integrate to 0 over symmetric intervals.

- Forgetting that "area under" includes the strip down to the x-axis; if the question asks "area between curve and a horizontal line $y = k$", use $\int (k - f(x))\,dx$ (assuming $k > f$).
- Using wrong limits in horizontal strips. For $y^2 = 4ax$ to the line $y = 3$, the limits are $y = 0$ to $y = 3$, not $x = 0$ to $x =$ some value.
- Computing only the first-quadrant area and forgetting the symmetry multiplier.
- Treating the parabola as a function in both x and y simultaneously: $y^2 = 4ax$ does not give a single y for each x unless we specify the branch.

2.5 Key formulas & theorems

Formula	Statement	NCERT page
Vertical-strip area	$A = \int_a^b f(x)\,dx$	293
Horizontal-strip area	$A = \int_c^d g(y)\,dy$	293
Elementary area (vertical)	$dA = y\,dx$	292
Elementary area (horizontal)	$dA = x\,dy$	293
Below-axis correction	$A =$	$\int_a^b f(x)\,dx$
Crossing-axis area	$A =$	A_1
Circle area	πa^2	294
Ellipse area	πab	295
Quarter circle	$(\pi a^2)/4$	294
Integral $\int\sqrt{(a^2-x^2)}\,dx$	$(x/2)\sqrt{(a^2-x^2)} + (a^2/2)\sin^{-1}(x/a) + C$	294
First-quadrant ellipse	$\pi ab/4$	295
Parabola $y^2 = 4ax$ slice	$x = y^2/(4a)$	296
Area under line $y = 3x + 2$	13/3 (on $[-1, 1]$)	297
Area under cos x on $[0, 2\pi]$	4	297
Area between two curves	$\int(\text{top} - \text{bottom})\,dx$	implicit
Area between two y-curves	$\int(\text{right} - \text{left})\,dy$	implicit
Symmetry about y-axis	$2\int_0^a f\,dx$	294
Symmetry about both axes	$4\int_0^a f\,dx$ (first quadrant)	294
Quarter ellipse (vertical)	$(b/a)\int_0^a \sqrt{(a^2-x^2)}\,dx$	295
Quarter ellipse (horizontal)	$(a/b)\int_0^b \sqrt{(b^2-y^2)}\,dy$	296
Sign convention	$\int_a^b f\,dx \geqslant 0$ if $f \geqslant 0$	293
Riemann interpretation	$A = \lim$ of $\Sigma\, f(x_i)\,\Delta x$	292
Linearity of area	$A(R_1 \cup R_2) = A(R_1) + A(R_2)$ (disjoint)	294

Formula	Statement	NCERT page
Geometric area is non-negative		∫

2.6 Solved examples (NCERT-grounded)

Example A (NCERT Example 1, p. 294). Area enclosed by $x^2 + y^2 = a^2$.

Step 1 — symmetry: total area = 4 × (first-quadrant area). Step 2 — first-quadrant integral: 4 ∫_0^a √(a² − x²) dx. Step 3 — evaluate: using standard integral = 4·(a²·π/4) = **πa²**.

Example B (NCERT Example 2, p. 295). Area enclosed by $x^2/a^2 + y^2/b^2 = 1$.

Step 1 — symmetry: 4 × (first-quadrant area). Step 2 — substitute y = (b/a)√(a² − x²): 4 ∫_0^a (b/a)√(a² − x²) dx. Step 3 — evaluate: (4b/a)·(a²·π/4) = **πab**.

Example C (NCERT Misc. Example 3, p. 297). Area bounded by y = 3x + 2, x-axis, x = −1, x = 1.

Step 1 — find crossing: 3x + 2 = 0 ⇒ x = −2/3. Step 2 — split and integrate: |∫{−1}^{−2/3}(3x+2)dx| + ∫(3x+2)dx. }^{1 Step 3 — compute: 1/6 + 25/6 = **13/3**.

Example D (NCERT Misc. Example 4, p. 297). Area bounded by y = cos x, [0, 2π].

Step 1 — sign analysis: cos > 0 on [0, π/2] ∪ [3π/2, 2π]; cos < 0 on [π/2, 3π/2]. Step 2 — compute each piece: ∫0^{π/2} cos x dx = 1; |∫ cos x dx = 1. }^{3π/2} cos x dx| = 2; ∫_{3π/2}^{2π Step 3 — sum: 1 + 2 + 1 = **4**.

Example E (NCERT Exercise 8.1 Q4, p. 296). Area bounded by $y^2 = 4x$, y-axis, line y = 3.

Step 1 — horizontal strip: x = y²/4; dA = x dy. Step 2 — integrate: ∫_0^3 (y²/4) dy = (1/4)·[y³/3]_0^3 = 27/12. Step 3 — simplify: 27/12 = **9/4**.

🎯 Practice MCQs

📊 PYQ Alignment

CUET (UG) Mathematics has carried 1–2 area-application MCQs every year since 2023, almost always testing one of three things: the standard areas of the circle (`πa²`) and ellipse (`πab`), area under a line/parabola between given ordinates (often requiring the absolute-value handling for portions below the x-axis), or the choice between vertical and horizontal strips for a parabola such as `y² = 4ax` . Expect direct-formula questions and at least one statement/assertion-style item built around the "below the x-axis ⇒ take modulus" rule.