FREE EDITION · NOTES + 3 SAMPLE MCQS

CUET · MATHEMATICS · CLASS XII · CODE 319

Continuity and Differentiability

CUET unit: Continuity and Differentiability

By **UniDrill** · NCERT-grounded study material

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📌 Snapshot

- Building on Class XI differentiation: a function is **continuous** when its limit equals its functional value, and **differentiable** when the limit defining the derivative exists; Theorem 3 links the two.
- Builds the **algebra of continuous functions** (sum, difference, product, quotient, composition) and the **algebra of derivatives** (sum, product, quotient, chain rule).
- Standard derivatives of **inverse trigonometric**, **exponential**, and **logarithmic** functions follow, plus **logarithmic differentiation** for forms like `[u(x)]^v(x)`.
- **Implicit differentiation**, **parametric differentiation**, and **second-order derivatives** are covered through worked examples.
- CUET regularly tests continuity-checking at piecewise junctions, chain-rule computations, and standard derivatives of inverse-trig, exponential, and log functions.

📖 Detailed Notes

2.1 Core concepts

- **Continuity at a point**: A real function `f` is continuous at `c` in its domain if `lim_{x→c} f(x) = f(c)`; equivalently, the left-hand limit, right-hand limit, and `f(c)` all exist and are equal (NCERT §5.2, p. 105). All three conditions are needed; relaxing any one breaks continuity.
- **Continuity on domain**: `f` is continuous if it is continuous at every point of its domain; on `[a, b]`, continuity at endpoints uses only one-sided limits — `lim_{x→a+} f(x) = f(a)` and `lim_{x→b-} f(x) = f(b)` (NCERT §5.2, p. 107).
- **Standard continuous functions** established in worked examples: constant function, identity `f(x)=x`, polynomial, rational (where denominator ≠ 0), `|x|`, `sin x`, `cos x`, and `tan x` wherever defined (NCERT §5.2, pp. 106–115). These serve as "building blocks" — combinations of them are continuous wherever the algebra is legal.
- **Greatest integer function** `f(x) = [x]` is discontinuous at every integer and continuous everywhere else (NCERT §5.2, Example 15, p. 112). The jumps are of size 1 at each integer.

- **Algebra of continuous functions (Theorem 1)**: if f and g are continuous at c, then $f \pm g$, $f \cdot g$, and f/g (when $g(c) \neq 0$) are continuous at c (NCERT §5.2.1, p. 113).
- **Composition (Theorem 2)**: if g is continuous at c and f is continuous at $g(c)$, then $f \circ g$ is continuous at c (NCERT §5.2.1, p. 115). The composition theorem powers most "is this function continuous?" arguments.
- **Derivative definition**: $f'(c) = \lim_{h \to 0} [f(c+h) - f(c)]/h$ when this limit exists; f is differentiable at c iff the left and right derivatives are finite and equal (NCERT §5.3, p. 118).
- **Theorem 3 (Differentiability ⇒ Continuity)**: every differentiable function is continuous; the converse fails — $f(x) = |x|$ is continuous but not differentiable at $x = 0$ (NCERT §5.3, p. 120). This is a one-way implication of singular importance for CUET.
- **Algebra of derivatives**: $(u \pm v)' = u' \pm v'$, product rule $(uv)' = u'v + uv'$, quotient rule $(u/v)' = (u'v - uv')/v^2$ for $v \neq 0$ (NCERT §5.3, p. 119).
- **Chain rule (Theorem 4)**: for $f = v \circ u$ with $t = u(x)$, $df/dx = (dv/dt)\cdot(dt/dx)$; extended to triple compositions as $(dw/ds)(ds/dt)(dt/dx)$ (NCERT §5.3.1, p. 121). The chain rule is the workhorse of all composite-function derivatives.
- **Implicit differentiation**: when y is given implicitly by a relation in x and y, differentiate both sides w.r.t. x treating y as a function of x, then solve for dy/dx (NCERT §5.3.2, pp. 122–123).
- **Derivatives of inverse trig functions**: $d/dx(\sin^{-1} x) = 1/\sqrt{(1 - x^2)}$, $d/dx(\cos^{-1} x) = -1/\sqrt{(1 - x^2)}$, $d/dx(\tan^{-1} x) = 1/(1 + x^2)$; $\sin^{-1}$ and $\cos^{-1}$ derivatives valid on $(-1, 1)$, $\tan^{-1}$ valid on R (NCERT §5.3.3, p. 124).
- **Exponential & log derivatives (Theorem 5)**: $d/dx(e^x) = e^x$ and $d/dx(\log x) = 1/x$ for $x > 0$; in this chapter $\log x$ denotes natural log (base e) (NCERT §5.4, p. 129). The exponential is the unique function (up to constant multiple) equal to its own derivative.
- **Derivative of a^x** (Example 28): $d/dx(a^x) = a^x \log a$ for positive constant a; obtained by writing $a^x = e^{x \log a}$ or by logarithmic differentiation (NCERT §5.5, p. 131).
- **Change-of-base formula**: $\log_a p = (\log_b p)/(\log_b a)$; applied in Example 39 to differentiate $\log_7(\log x)$ as $(\log(\log x))/\log 7$ (NCERT §5.4, p. 128; Miscellaneous Example, p. 141).
- **Logarithmic differentiation**: for $y = [u(x)]^{v(x)}$ (with $u(x) > 0$), take $\log y = v(x) \log u(x)$ then differentiate; standard technique for x^x, $x^{\sin x}$, etc. (NCERT §5.5, p. 130). Without taking logs first, the standard power-rule and exponential-rule cannot be applied directly.

- **Parametric differentiation**: if $x = f(t)$, $y = g(t)$, then $dy/dx = (dy/dt)/(dx/dt) = g'(t)/f'(t)$ whenever $f'(t) \neq 0$ (NCERT §5.6, p. 135). Used widely in curves like the cycloid and ellipse.
- **Second-order derivative**: $d^2y/dx^2 = d/dx(dy/dx)$, also denoted $f''(x)$, y'', y_2, or D^2y; higher orders defined analogously (NCERT §5.7, p. 137). Second derivatives feature in concavity, inflection, and physical interpretations like acceleration.
- These derivatives and rules are foundational for Class XII Applications of Derivatives (lemh106), Integrals (lemh107) and beyond — they recur throughout the calculus syllabus.
- The historical thread: continuity (Cauchy, Weierstrass) and differentiability (Newton, Leibniz) were unified into modern analysis through $\varepsilon-\delta$ definitions. NCERT uses the intuitive limit-based definitions rather than full $\varepsilon-\delta$ rigour, sufficient for Class XII problem solving.
- A subtle observation: every polynomial is everywhere differentiable; every rational function is differentiable on its domain; every trigonometric function is differentiable on its domain; every exponential is differentiable everywhere on R. Standard functions are well-behaved; trouble arises mainly at piecewise junctions, isolated points (like |x| at 0), or where denominators vanish.
- Newton's and Leibniz's notations both appear: dy/dx (Leibniz) and f'(x) (Lagrange) are the two main systems. Both denote the same derivative; CUET uses Lagrange notation more often in MCQ stems.
- Second-derivative significance: $d^2y/dx^2 > 0 \Rightarrow$ concave up; $< 0 \Rightarrow$ concave down; $= 0$ with change of sign $\Rightarrow$ inflection point. These ideas explicitly enter the next chapter on Applications of Derivatives.

2.2 Definitions to memorise

Term	Definition	Page
Continuity at c	$\lim_{x\to c} f(x) = f(c)$	105
Continuous function	Continuous at every point of domain	107
Point of discontinuity	Where f fails continuity	105
Left-hand limit	$\lim_{x\to c\ } f(x)$	105
Right-hand limit	$\lim_{x\to c\ } f(x)$	105
Derivative	$f'(c) = \lim_{h\to 0} (f(c+h) - f(c))/h$	118
Differentiable at c	Left, right derivatives finite and equal	119
Chain rule	$(v\circ u)' = v'(u)\cdot u'$	121
Product rule	$(uv)' = u'v + uv'$	119
Quotient rule	$(u/v)' = (u'v - uv')/v^2$	119

Term	Definition	Page
Sum rule	(u + v)' = u' + v'	119
Implicit differentiation	Treat y as fn of x; differentiate; solve	122
Logarithmic differentiation	log y = v(x) log u(x)	130
Parametric form	dy/dx = (dy/dt)/(dx/dt)	135
Second derivative	$d^2y/dx^2 = d/dx(dy/dx)$	137
Algebra of continuous functions	f±g, fg, f/g continuous	113
Composition theorem	f∘g continuous if pieces are	115
Differentiability ⇒ continuity	Theorem 3	120
Counter-example		x
Natural log	log x = ln x in this chapter	128
Change of base	log_a p = log p / log a	128
Standard derivative x^n	n x^(n−1)	119
Derivative of e^x	e^x	129
Derivative of log x	1/x	129
Derivative of a^x	a^x log a	131

Standard derivatives summarised (Summary box, p. 146 and Table 5.3, p. 119):

f(x)	f'(x)
x^n	$n x^{n-1}$
sin x	cos x
cos x	−sin x
tan x	$\sec^2 x$
cot x	$-\text{cosec}^2 x$
sec x	sec x tan x
cosec x	−cosec x cot x
$\sin^{-1} x$	$1/\sqrt{(1 - x^2)}$
$\cos^{-1} x$	$-1/\sqrt{(1 - x^2)}$
$\tan^{-1} x$	$1/(1 + x^2)$
$\cot^{-1} x$	$-1/(1 + x^2)$
$\sec^{-1} x$	$1/(\|x\|\sqrt{(x^2 - 1)})$
$\text{cosec}^{-1} x$	$-1/(\|x\|\sqrt{(x^2 - 1)})$
e^x	e^x

$f(x)$	$f'(x)$
$\log x$	$1/x$
a^x	$a^x \log a$
$\log_a x$	$1/(x \log a)$

2.3 Diagrams / processes to remember

- **Fig 5.1 (p. 104)** – Step function with jump at $x = 0$: shows a function that cannot be drawn without lifting the pen, motivating the visual idea of discontinuity.
- **Fig 5.3 (p. 109)** – Graph of $f(x) = 1/x$: illustrates that left-hand limit is $-\infty$ and right-hand limit is $+\infty$ at $x = 0$, neither being a real number.
- **Fig 5.5 (p. 110)** – Piecewise function with isolated point of discontinuity at $x = 1$: typifies the "jump + redefined value" trap.
- **Fig 5.8 (p. 112)** – Graph of the greatest integer function: step-like staircase, discontinuous at every integer.
- **Figs 5.9–5.11 (pp. 125–128)** – Comparison of polynomial growth x^n vs exponential 10^x, plot of $y = b^x$, and $y = e^x$ and $y = \ln x$ as mirror images across $y = x$.
- **Process — check continuity at point c:** (i) compute lim_{x→c } f(x); (ii) compute lim_{x→c } f(x); (iii) check both equal; (iv) compare with f(c). If all three agree, f is continuous at c.
- **Process — find k so f is continuous:** equate the relevant one-sided limit to f(c) and solve for the unknown constant.
- **Process — differentiate composite via chain rule:** identify outer and inner functions; differentiate outer (keeping inner intact), multiply by derivative of inner.
- **Process — logarithmic differentiation:** take ln of both sides; differentiate implicitly; solve for dy/dx; multiply by y to express explicitly.
- **Process — parametric dy/dx:** find dx/dt and dy/dt; divide; ensure dx/dt ≠ 0 at the point of interest.

2.4 Common confusions / NTA trap points

- "Continuous ⇒ differentiable" is **false**. The standard counter-example is $f(x) = |x|$ at $x = 0$: left derivative is -1, right derivative is $+1$, so the function is continuous but not differentiable there (p. 120).
- For piecewise definitions, you must check **both** that one-sided limits agree **and** that the common value equals $f(c)$. Example 4 ($f(x) = x^3 + 3$ if $x \neq 0$, $f(0) = 1$) fails the second condition even though the limit exists (p. 107).
- $\lim_{x\to0^+} 1/x = +\infty$ and $\lim_{x\to0^-} 1/x = -\infty$ — but $\pm\infty$ are **not** real numbers, so these limits do **not** exist as real numbers (p. 109).

- In this chapter $\log x$ means **natural** log (base e), not base 10. So $d/dx(\log x) = 1/x$, not $1/(x \ln 10)$ (p. 128).
- For $d/dx(a^x) = a^x \log a$, the $\log a$ is $\ln a$; students wrongly write a^x or $x a^{x-1}$. The correct derivation uses $a^x = e^{x \log a}$ (Example 28, p. 131).
- The derivative of $\sin^{-1} x$ is only valid on the **open** interval $(-1, 1)$; at $x = \pm 1$ the derivative is not defined because $\cos y = 0$ there (p. 124).
- For $y = x^x$, neither the power rule ($x \cdot x^{x-1}$) nor the exponential rule ($x^x \log x$) is correct in isolation; use logarithmic differentiation to get $x^x(1 + \log x)$.
- Forgetting to differentiate the inner function in chain rule — $d/dx[\sin(x^2)] = 2x \cos(x^2)$, not $\cos(x^2)$.
- Mis-applying quotient rule with sign error — denominator squared is always v^2, not v.
- Confusing implicit and parametric: implicit has y appearing in a relation with x; parametric has x and y both as functions of a third variable t.
- Forgetting to keep dy/dx on one side when doing implicit differentiation. After differentiating, collect dy/dx terms on LHS and solve.
- Stopping at first derivative when second is asked. Always differentiate again.
- Confusing log_a x with ln x; the conversion is log_a x = ln x / ln a.
- Treating the derivative of a constant as anything other than 0. Constants vanish under differentiation.
- Forgetting that the chain rule applies even to nested compositions of three or more functions; each level contributes a factor.
- Mis-using the quotient rule by reversing numerator and denominator. The correct numerator is u'v – uv', NOT v'u – vu'.
- Skipping the absolute-value in the derivatives of sec ^{1} x and cosec ^{1} x: they have |x| in the denominator because the range of these functions is restricted to $[0, \pi] - \{\pi/2\}$ and $[-\pi/2, \pi/2] - \{0\}$ respectively.
- Computing the derivative of (sin x)^x using the power rule alone — this is a "function to function" power, requiring logarithmic differentiation.
- Misinterpreting "find dy/dx" in parametric form as differentiating y w.r.t. t; the correct interpretation is the ratio (dy/dt)/(dx/dt).
- Forgetting that continuity at a single point does not imply continuity on an interval — verifying continuity at one point is necessary but not sufficient.
- Mishandling derivative of inverse trigonometric functions when the input is not x but a function of x (e.g., sin ^{1}(2x – 1) requires chain rule: derivative is $2/\sqrt{(1 - (2x - 1)^2)}$).
- Treating |x| as differentiable everywhere; it is differentiable on R \ {0} but not at 0 itself, where the corner has slopes ±1.
- Confusing the chain rule's "outer × inner" with reverse order — the outer function's derivative comes first.

- Forgetting that the product rule symbol u'v + uv' is symmetric; some students drop one of the two terms.

2.5 Key formulas & theorems

Formula	Statement	NCERT page
Continuity at c	lim f(x) = f(c)	105
Derivative definition	f'(c) = lim (Δf/h)	118
Theorem 3	Differentiable ⇒ continuous	120
Algebra (sum)	(u ± v)' = u' ± v'	119
Algebra (product)	(uv)' = u'v + uv'	119
Algebra (quotient)	(u/v)' = (u'v − uv')/v^2	119
Chain rule	(v∘u)'(x) = v'(u(x)) u'(x)	121
d/dx x^n	n x^(n−1)	119
d/dx sin x	cos x	119
d/dx cos x	−sin x	119
d/dx tan x	$\sec^2$ x	119
d/dx sin ^{1} x	1/√(1 − x^2)	124
d/dx cos ^{1} x	−1/√(1 − x^2)	124
d/dx tan ^{1} x	1/(1 + x^2)	124
d/dx e^x	e^x	129
d/dx log x	1/x	129
d/dx a^x	a^x log a	131
d/dx log_a x	1/(x log a)	132
Logarithmic differentiation	log y = v log u, differentiate	130
Parametric dy/dx	(dy/dt)/(dx/dt)	135
Second derivative	d/dx(dy/dx)	137
d/dx (x^x)	x^x(1 + log x)	133
d/dx sec x	sec x tan x	119
d/dx cosec x	−cosec x cot x	119
d/dx cot x	−cosec^2 x	119

2.6 Solved examples (NCERT-grounded)

Example A (NCERT Example 10, p. 110). Is f continuous at x = 1, where f(x) = x + 2 for x ⩽ 1 and f(x) = x − 2 for x > 1?

Step 1 — LHL: lim_{x→1 }(x + 2) = 3. Step 2 — RHL: lim_{x→1 }(x − 2) = −1. Step 3 — compare: LHL ≠ RHL ⇒ **discontinuous at x = 1**.

Example B (NCERT Example 21, p. 121). Differentiate $\sin(x^2)$.

Step 1 — let $t = x^2$: $y = \sin t$. Step 2 — chain rule: $dy/dx = (dy/dt)(dt/dx) = \cos t \cdot 2x$. Step 3 — substitute: **$2x \cos(x^2)$**.

Example C (NCERT Example 28, p. 131). Find d/dx (a^x).

Step 1 — log both sides of $y = a^x$: log y = x log a. Step 2 — differentiate: (1/y)(dy/dx) = log a. Step 3 — solve: dy/dx = y log a = **a^x log a**.

Example D (NCERT Example 30, p. 133). Find dy/dx for $y = x^x$, x > 0.

Step 1 — log both sides: log y = x log x. Step 2 — differentiate: (1/y)(dy/dx) = log x + x(1/x) = 1 + log x. Step 3 — solve: dy/dx = **x^x(1 + log x)**.

Example E (NCERT Example 32, p. 135). $x = at^2$, y = 2at. Find dy/dx.

Step 1 — dx/dt: 2at. Step 2 — dy/dt: 2a. Step 3 — divide: dy/dx = 2a/(2at) = **1/t**.

Example F (NCERT Example 35, p. 138). Find d^2y/dx^2 for $y = x^3 + \tan x$.

Step 1 — first derivative: $dy/dx = 3x^2 + \sec^2 x$. Step 2 — differentiate term-by-term: $d/dx(3x^2) = 6x$; $d/dx(\sec^2 x) = 2 \sec x \cdot (\sec x \tan x) = 2 \sec^2 x \tan x$. Step 3 — combine: d^2y/dx^2 = **$6x + 2 \sec^2 x \tan x$**.

Example G (Continuity at a junction). Find k so that $f(x) = kx^2$ for $x \leqslant 2$ and f(x) = 3 for x > 2 is continuous at x = 2.

Step 1 — LHL: lim_{x→2 }(kx^2) = 4k. Step 2 — RHL: lim_{x→2 }(3) = 3. Step 3 — equate: 4k = 3 ⇒ **k = 3/4**.

🎯 Practice MCQs

📊 PYQ Alignment

This chapter is one of the highest-yielding in CUET Mathematics — across CUET (UG) 2023–25 papers, roughly 10–12 MCQs per year are pegged to it. Questions typically test continuity at piecewise junctions (find `k / a, b` so the function is continuous), direct chain-rule and product-rule computation, derivatives of inverse-trig forms, logarithmic differentiation of `[u(x)]^{v(x)}` types, parametric `dy/dx`, and one or two second-order derivative items.