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CUET · MATHEMATICS · CLASS XII · CODE 319

Differential Equations

CUET unit: Differential Equations

By **UniDrill** · NCERT-grounded study material

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📌 Snapshot

- An ordinary differential equation involves derivatives of a dependent variable with respect to one independent variable.
- Establishes the twin descriptors of every ODE — **order** (highest derivative present) and **degree** (highest power of that derivative, only when the equation is a polynomial in derivatives).
- Distinguishes **general solution** (contains as many arbitrary constants as the order) from **particular solution** (constants fixed by initial conditions).
- Develops three solution techniques for first-order first-degree ODEs: **variables separable**, **homogeneous** (substitution $y = vx$ or $x = vy$), and **linear** (integrating factor $e^{\int P\,dx}$).
- Applies these methods to geometric problems (curves through a point), growth/decay (bank principal, bacteria culture, balloon inflation). CUET draws factual, formula-recall and one-line-solve MCQs from every section.

📖 Detailed Notes

2.1 Core concepts

- An equation involving the derivative(s) of a dependent variable with respect to an independent variable is a **differential equation**; if only one independent variable is involved, it is an **ordinary differential equation (ODE)**, otherwise a partial differential equation — this chapter studies ODEs only (NCERT §9.2, p. 300–301). Examples introduction include $dy/dx = e^x$, $d^2y/dx^2 + y = 0$, and $xy(d^2y/dx^2) + x(dy/dx)^2 - y(dy/dx) = 0$.
- Standard derivative notation used throughout: $dy/dx = y'$, $d^2y/dx^2 = y''$, $d^3y/dx^3 = y'''$, and y⍰ for the nth-order derivative (NCERT §9.2 Note, p. 301). The prime notation is preferred in compact statements and is the form CUET uses in MCQ stems.
- **Order** of a differential equation is the order of the highest-order derivative appearing in it; e.g. $dy/dx = e^x$ has order 1, $d^2y/dx^2 + y = 0$ has order 2, $d^3y/dx^3 + x^2(d^2y/dx^2)^3 = 0$ has order 3 (NCERT §9.2.1, p. 301–302). Order is always a positive integer and is determined unambiguously by inspection.
- **Degree** is defined **only if** the equation is a polynomial in the derivatives $y', y'', y''', \cdots$; if defined, it is the highest (positive integral) power of the highest-order derivative

(NCERT §9.2.2, p. 302). The qualifier "polynomial in derivatives" is the single most common source of conceptual errors.

- Equations such as (dy/dx) + sin(dy/dx) = 0 are **not** polynomial in y′, so their degree is **not defined**; order and degree (when defined) are always positive integers (NCERT §9.2.2 and Note, p. 302). Other examples of non-polynomial DEs include those involving e^(y′), log(y″), √(y′), or fractional powers of derivatives — none of these admit a degree.
- A **solution** is a function y = ϕ(x) which, when substituted into the equation, reduces L.H.S. = R.H.S.; the curve y = ϕ(x) is the **solution / integral curve** (NCERT §9.3, p. 304). Verification of a candidate solution is done by direct substitution.
- **General solution** contains as many arbitrary constants as the order of the equation; a **particular solution** is obtained by giving definite values to those constants (NCERT §9.3, p. 305). A first-order DE has a one-parameter family of solutions; a second-order DE has a two-parameter family; etc.
- **Formation of a DE** whose general solution is given: differentiate the family enough times to eliminate every arbitrary constant; an n-parameter family produces an n-th-order DE (concept used in Exercise 9.2 Q11–Q12 and miscellaneous examples).
- **Variables-separable form** dy/dx = h(y)·g(x): rewrite as dy/h(y) = g(x) dx and integrate both sides to get H(y) = G(x) + C (NCERT §9.4.1, p. 306–307). The technique works whenever the right-hand side factors into a function of y times a function of x.
- Worked example: dy/dx = (x + 1)/(2 − y) separates to (2 − y) dy = (x + 1) dx and integrates to $x^2 + y^2 + 2x - 4y + C = 0$ (NCERT Example 4, p. 307). The implicit form is acceptable as the "general solution".
- Application — bank principal growing continuously at 5% per year obeys dP/dt = P/20, a first-order separable DE giving P = 1000 e^(t/20); doubling time t = 20 $\log_e 2$ ≈ 13.86 years (NCERT Example 9, p. 310).
- **Homogeneous function**: $F(\lambda x, \lambda y) = \lambda^n F(x, y)$ for some constant n called the degree of homogeneity; the corresponding DE dy/dx = F(x, y) is **homogeneous** when F is homogeneous of degree **zero**, i.e. F can be written purely in terms of the ratio y/x as g(y/x) (NCERT §9.4.2, p. 312–313).
- Standard substitution y = v·x (so dy/dx = v + x·dv/dx) reduces the equation to a variables-separable form ∫ dv/[g(v) − v] = ∫ dx/x + C (NCERT §9.4.2, p. 313–314). After integration, back-substitute v = y/x to obtain the general solution in (x, y).
- If the equation is written as dx/dy = h(x/y), use x = v·y instead, so dx/dy = v + y·dv/dy (NCERT §9.4.2 Note, p. 314). The "direction" of the substitution depends on whether dy/dx or dx/dy is more naturally expressed.
- **Linear differential equation** in y: dy/dx + Py = Q where P, Q are constants or functions of x only; multiplying both sides by the **integrating factor** I.F. = e^(∫P dx) makes the L.H.S. an exact derivative d/dx[y · I.F.] (NCERT §9.4.3, p. 322–323).

- Solution of a linear DE: y · (I.F.) = ∫ Q · (I.F.) dx + C (NCERT §9.4.3, p. 323). This three-step recipe — rewrite in standard form, compute I.F., integrate Q × I.F. — is the workhorse of CUET DE questions.
- Mirror form: $dx/dy + P_1x = Q_1$ (P_1, Q_1 functions of y only) has I.F. = $e^{\int P_1\, dy}$ and solution x · (I.F.) = $\int Q_1 \cdot (\text{I.F.})\, dy + C$ (NCERT §9.4.3, p. 324). Use this when the equation is more naturally linear in x.
- Worked examples: dy/dx − y = cos x → P = −1, I.F. = e^{-x}, and y = [(sin x − cos x)/2] + C e^x (Example 14, p. 324); x dy/dx + 2y = x^2 rewrites as dy/dx + (2/x)y = x with I.F. = x^2, solution y = $x^2/4 + C x^{-2}$ (Example 15, p. 325).
- Many physical, biological, and economic laws are most naturally expressed as differential equations — Newton's second law, radioactive decay, Newton's law of cooling, RC-circuit voltage decay, logistic growth — so these techniques generalise far beyond mathematics (NCERT §9.4 closing remarks, p. 326).

2.2 Definitions to memorise

Term	Definition	Page
Differential equation	Equation involving derivative(s) of a dependent variable w.r.t. independent variable(s)	300–301
Ordinary DE	DE with only **one** independent variable	301
Order	Order of the highest derivative present	301
Degree	Highest positive-integral power of the highest-order derivative, only if DE is polynomial in derivatives	302
Polynomial in derivatives	DE where every derivative appears with a non-negative integer power, no transcendental wrapping	302
Solution / integral curve	Function y = ϕ(x) satisfying the DE; graph is integral curve	304
General solution (primitive)	Solution containing arbitrary constants equal in number to the order	305
Particular solution	Solution obtained by fixing all arbitrary constants	305
Initial condition	Value of y (or its derivatives) prescribed at a specific x	305
Variables-separable DE	dy/dx = g(x) · h(y); solved by ∫ dy/h(y) = ∫ g(x) dx + C	306–307
Homogeneous function of degree n	F(λx, λy) = λ^n F(x, y) for every non-zero λ	312
Homogeneous DE	dy/dx = F(x, y) with F homogeneous of degree zero, i.e. F(x, y) = g(y/x)	313
Substitution y = vx	Reduces homogeneous DE to separable form in (v, x)	313
Substitution x = vy	Used when DE is dx/dy = h(x/y)	314

Term	Definition	Page
Linear DE (in y)	$dy/dx + P(x)\,y = Q(x)$	322
Linear DE (in x)	$dx/dy + P_1(y)\,x = Q_1(y)$	324
Integrating Factor (I.F.)	$e^{\int P\,dx}$ for the y-linear form; $e^{\int P_1\,dy}$ for the x-linear form	323–324
Standard form (linear y-DE)	$dy/dx + P(x)\,y = Q(x)$ — necessary before computing I.F.	322
Formation of DE	Procedure of differentiating a family with n constants n times to eliminate them	304
Order = number of constants	Order of the DE = number of arbitrary constants in its primitive	305
Continuous growth model	$dP/dt = kP \Rightarrow P = P_0 e^{kt}$	310
Exponential decay model	$dy/dt = -ky \Rightarrow y = y_0 e^{-kt}$	310
Newton's law of cooling	$dT/dt = -k(T - T_0)$	326
Singular solution	A solution not obtainable from the general solution by any choice of constant	305

2.3 Diagrams / processes to remember

- **Three-step recipe for a linear DE (NCERT §9.4.3, p. 324):** (i) write the equation in the form $dy/dx + Py = Q$, (ii) compute I.F. $= e^{\int P\,dx}$, (iii) write $y \cdot (\text{I.F.}) = \int Q \cdot (\text{I.F.})\,dx + C$. Skipping step (i) — failing to divide through to make the y′ coefficient 1 — is the single biggest source of wrong I.F.s.
- **Substitution flow for a homogeneous DE (NCERT §9.4.2, p. 313–314):** put $y = vx \rightarrow dy/dx = v + x\cdot dv/dx \rightarrow$ original equation becomes $x\cdot dv/dx = g(v) - v \rightarrow$ separate variables → integrate → replace v by y/x. Forgetting the chain rule on dy/dx (writing $dy/dx = v$ instead of $v + x\cdot dv/dx$) is a recurring error.
- **Recognition test for homogeneity (NCERT p. 312):** in the candidate function F(x, y), replace x by λx and y by λy and simplify; if every λ cancels (i.e. λ^0 remains), the DE is homogeneous of degree 0. If λ^n remains with $n \neq 0$, F is homogeneous of degree n but the DE itself is not "a homogeneous DE"'s strict sense.
- **Recognition test for linearity:** rewrite the DE so that y′ has coefficient 1; if what remains is y′ + (function of x only) · y = (function of x only), the equation is linear in y. If the equation contains y^2 or sin y or e^y, it is non-linear.
- **Growth/decay template (NCERT Example 9, p. 309–310):** "rate proportional to amount" $\Rightarrow dP/dt = kP \Rightarrow P = P_0 e^{kt}$; used for bank principal, bacterial culture (Exercise 9.3 Q20–Q22), and balloon volume (Q19). Always identify P_0 from the initial condition before applying.

- **Formation-of-DE flow:** given $y = f(x, c_1, c_2, \cdots, c_n)$, differentiate n times to obtain n + 1 equations; eliminate the n constants among them; the surviving equation is the required DE of order n.
- **Verification flow:** to check whether $y = \phi(x)$ solves a given DE, compute $\phi'(x)$, $\phi''(x)$, ⋯ as needed, substitute into LHS, and confirm it equals RHS identically in x.
- **General-solution check:** count the arbitrary constants; they must equal the order of the equation. A "first-order" solution with two constants signals algebraic error.

2.4 Common confusions / NTA trap points

- **Degree-not-defined trap.** If the DE contains sin(y′), e^(y′), log(y′), etc., the equation is **not polynomial** in derivatives — degree is "not defined," even though order is perfectly defined (NCERT §9.2.2, p. 302; Q11 of Exercise 9.1, p. 303).
- **Highest-power vs highest-order.** Degree is the power of the **highest-order** derivative, not the largest power appearing anywhere. In $(y''')^2 + (y'')^3 + (y')^4 + y^5 = 0$, order = 3 and degree = 2, not 4 or 5 (NCERT Exercise 9.1 Q6, p. 303).
- **Number of arbitrary constants.** A general solution of an n-th order DE has exactly n arbitrary constants; a **particular** solution has **zero** constants (NCERT Exercise 9.2 Q11–Q12, p. 306).
- **Homogeneity check.** F(x, y) = sin x + cos y is **not** homogeneous (cannot be written as $\lambda^n F(x, y)$); whereas $y^2 + 2xy$, $2x - 3y$ and cos(y/x) are homogeneous of degrees 2, 1, 0 respectively (NCERT §9.4.2, p. 312). Only degree-0 functions give "homogeneous DEs" in the textbook's sense.
- **Wrong substitution direction.** If the DE comes naturally as dx/dy = h(x/y), substitute x = vy, **not** y = vx (NCERT §9.4.2 Note, p. 314; Exercise 9.4 Q16, p. 321).
- **Integrating factor sign.** For dy/dx − y = cos x, P = −1, so I.F. = e^(−x), not e^x (NCERT Example 14, p. 324). Students often forget the sign of P after rewriting in standard form.
- **Failure to normalise y′ coefficient.** In $x(dy/dx) - y = 2x^2$, one must first divide by x to read off P = −1/x; reading off P = −1 from the un-divided form gives a wrong I.F.
- **Confusing linear with first-degree.** "Linear" requires the dependent variable and its derivatives to appear linearly. $dy/dx = y^2$ is first-degree (degree 1 in y′) but **not** linear.
- **Mistaking dy/dx = e^(x+y) for non-separable.** It separates: $e^{(x+y)} = e^x \cdot e^y$, so the equation is variables-separable (Exercise 9.3 Q23).
- **Integration constant placement.** Place +C on the right side only, after both integrations; placing it before back-substitution often produces an algebraically equivalent but exam-marker-unfriendly form.
- **Failure to back-substitute v = y/x.** After solving the separated form in (v, x), some students forget to write the answer in (x, y).

- **Treating "particular solution" as having one arbitrary constant.** Once initial conditions are applied, all constants are determined; the particular solution is constant-free.

2.5 Key formulas & theorems

Formula	Statement	NCERT page
Notation y′, y″, ⋯	$y' = dy/dx$, $y'' = d^2y/dx^2$, etc.	301
Order definition	Order = highest derivative present	301
Degree definition	Highest power of highest-order derivative, polynomial-in-derivatives required	302
Number-of-constants rule	n-th order DE ⇒ n arbitrary constants in general solution	305
Variables separable form	$dy/h(y) = g(x)\,dx$	307
Variables separable solution	$\int dy/h(y) = \int g(x)\,dx + C$	307
Homogeneous function	$F(\lambda x, \lambda y) = \lambda^n F(x, y)$	312
Homogeneous DE (definition)	$dy/dx = g(y/x)$	313
Substitution for homogeneous DE	$y = vx \Rightarrow dy/dx = v + x\,dv/dx$	313
Separated form (homogeneous)	$dv/[g(v) - v] = dx/x$	313
Mirror substitution	$x = vy$ if equation is $dx/dy = h(x/y)$	314
Linear DE form (in y)	$dy/dx + P(x)\,y = Q(x)$	322
Integrating factor (in y)	I.F. = $e^{(\int P\,dx)}$	323
Linear DE solution	$y \cdot (\text{I.F.}) = \int Q \cdot (\text{I.F.})\,dx + C$	323
Linear DE form (in x)	$dx/dy + P_1(y)\,x = Q_1(y)$	324
Integrating factor (in x)	I.F. = $e^{(\int P_1\,dy)}$	324
Linear DE solution (in x)	$x \cdot (\text{I.F.}) = \int Q_1 \cdot (\text{I.F.})\,dy + C$	324
Continuous growth	$dP/dt = kP \Rightarrow P = P_0\,e^{(kt)}$	310
Doubling time (k = 1/20)	$t = 20 \log 2$	310
Verification	Substitute $y = \phi(x)$ and confirm LHS = RHS	304
Linearity test	y, y′, y″, ⋯ appear to first power, no products among them	322
Polynomial-in-derivatives test	No sin, e^(·), log, √ wrapping derivatives	302

Formula	Statement	NCERT page
$dy/dx = e^{(x+y)}$ separation	$e^{(-y)}\,dy = e^x\,dx$	312
Standard I.F. for $x y' - y = 2x^2$	I.F. = $1/x$	329
Standard I.F. for $(1-y^2)x' + yx = ay$	I.F. = $1/\sqrt{(1-y^2)}$	329

2.6 Solved examples (NCERT-grounded)

Example A (NCERT Example 4, p. 307). Solve $dy/dx = (x + 1)/(2 - y)$.

Step 1 — separate variables: $(2 - y)\,dy = (x + 1)\,dx$. Step 2 — integrate both sides: $2y - y^2/2 = x^2/2 + x + C_1$. Step 3 — rearrange: $x^2 + y^2 + 2x - 4y + C = 0$ where $C = -2C_1$. **Answer:** general solution in implicit form.

Example B (NCERT Example 9, p. 309–310). A bank pays continuously compounded interest at 5%; initial deposit ₹1000. Find amount after t years and doubling time.

Step 1 — write DE: $dP/dt = (5/100)\,P = P/20$. Step 2 — separate and integrate: $dP/P = dt/20 \Rightarrow \log P = t/20 + C$; using $P(0) = 1000$ gives $C = \log 1000$, so $P = 1000\,e^{(t/20)}$. Step 3 — doubling time: $2 = e^{(t/20)} \Rightarrow t = 20 \log 2 \approx 13.86$ years. **Answer:** $P(t) = 1000\,e^{(t/20)}$; doubling in $20 \ln 2$ years.

Example C (NCERT Example 14, p. 324). Solve $dy/dx - y = \cos x$.

Step 1 — identify P, Q: $P = -1$, $Q = \cos x$; I.F. = $e^{(\int -1\,dx)} = e^{(-x)}$. Step 2 — apply linear-solution formula: $y \cdot e^{(-x)} = \int \cos x \cdot e^{(-x)}\,dx + C$. Step 3 — evaluate integral: $\int e^{(-x)} \cos x\,dx = e^{(-x)}(\sin x - \cos x)/2$, so $y \cdot e^{(-x)} = e^{(-x)}(\sin x - \cos x)/2 + C \Rightarrow y = (\sin x - \cos x)/2 + C\,e^x$. **Answer:** $y = (\sin x - \cos x)/2 + C\,e^x$.

Example D (NCERT Example 15, p. 325). Solve $x\,dy/dx + 2y = x^2$.

Step 1 — divide by x: $dy/dx + (2/x)\,y = x$; here $P = 2/x$, $Q = x$. Step 2 — compute I.F.: I.F. $= e^{(\int 2/x\,dx)} = e^{(2 \log x)} = x^2$. Step 3 — integrate Q · I.F.: $y \cdot x^2 = \int x \cdot x^2\,dx + C = x^4/4 + C \Rightarrow y = x^2/4 + C/x^2$. **Answer:** $y = x^2/4 + C\,x^{-2}$.

Example E (Exercise 9.3 Q23, p. 312). Solve $dy/dx = e^{(x + y)}$.

Step 1 — factor: $e^{(x + y)} = e^x \cdot e^y$, so $dy/dx = e^x \cdot e^y$ — separable. Step 2 — separate and integrate: $e^{(-y)}\,dy = e^x\,dx \Rightarrow -e^{(-y)} = e^x - C$. Step 3 — rearrange: $e^x + e^{(-y)} = C$. **Answer:** $e^x + e^{(-y)} = C$.

🎯 Practice MCQs

📊 PYQ Alignment

Differential Equations is a high-yield CUET (UG) unit — typically 8–10 MCQs per paper across CUET 2023–25, drawn evenly from order/degree determination (1–2 Qs), recognising homogeneity or linear form (1–2 Qs), writing integrating factors (2 Qs), and one-line solves of variables-separable / linear DEs (3–4 Qs). The four NCERT in-text MCQs (Exercise 9.1 Q11–12, 9.2 Q11–12, 9.3 Q23, 9.4 Q16–17, 9.5 Q18–19) have repeatedly served as direct templates for CUET stems.