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CUET · MATHEMATICS · CLASS XII · CODE 319

Three Dimensional Geometry

CUET unit: Three Dimensional Geometry

By **UniDrill** · NCERT-grounded study material

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📌 Snapshot

- Establishes the language of lines in 3D space using direction cosines (DCs) and direction ratios (DRs), with the identity $l^2 + m^2 + n^2 = 1$ as the central fact.
- Builds vector and Cartesian equations of a line through (a) a point + direction and (b) two points; introduces the parametric form $r = a + \lambda b$.
- Develops formulas for the acute angle between two lines ($\cos\theta$ in terms of DRs and DCs) and the parallel/perpendicular conditions.
- Treats shortest distance between skew lines (using $b1 \times b2$) and between parallel lines (using $b \times (a2 - a1)$).
- In the rationalised reprint (2026-27), this chapter stops at lines in space; planes are NOT covered. CUET items from this chapter therefore focus on DCs/DRs, line equations, angle between lines and shortest distance.

📖 Detailed Notes

2.1 Core concepts

- A directed line L through the origin making angles α, β, γ with the positive x, y, z-axes has direction cosines $l = \cos\alpha$, $m = \cos\beta$, $n = \cos\gamma$ (NCERT §11.2, p. 377). These three numbers completely encode the direction of L in space.
- Reversing the direction of L replaces the angles by their supplements, so signs of all three DCs flip; a line in space has two sets of DCs unless taken as directed (NCERT §11.2, p. 377–378). The line itself is undirected; the DCs come with ± sign ambiguity.
- For a line not through the origin, draw a parallel line through O and read off its DCs — parallel lines share the same set of DCs (NCERT §11.2 Remark, p. 378). Direction is a translation-invariant property.
- Any three numbers proportional to l, m, n are called direction ratios (DRs); if a, b, c are DRs then $a = \lambda l$, $b = \lambda m$, $c = \lambda n$ for some nonzero λ (NCERT §11.2, p. 378). DRs are convenient because they often come out as integers, while DCs typically involve square roots.
- From $l^2 + m^2 + n^2 = 1$, the DCs in terms of DRs are $l = \pm a/\sqrt{(a^2+b^2+c^2)}$, $m = \pm b/\sqrt{(a^2+b^2+c^2)}$, $n = \pm c/\sqrt{(a^2+b^2+c^2)}$ (NCERT §11.2, p. 379). The ± reflects the direction-reversal ambiguity.

- The DCs of the line joining $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are $(x_2-x_1)/PQ$, $(y_2-y_1)/PQ$, $(z_2-z_1)/PQ$, where $PQ = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}$; the DRs may be taken as x_2-x_1, y_2-y_1, z_2-z_1 (NCERT §11.2.1, p. 379–380).
- A line is uniquely determined by either (i) a point and a direction, or (ii) two points (NCERT §11.3, p. 381). These two specifications underpin the two standard line equations.
- Vector equation of a line through point A (position vector a) parallel to b: $r = a + \lambda b$, $\lambda \in R$ (NCERT §11.3.1, p. 381–382). The parameter λ "slides" along the line; varying λ over R sweeps out the whole line.
- Writing $b = a\hat{i} + b\hat{j} + c\hat{k}$, the parametric form is $x = x_1+\lambda a$, $y = y_1+\lambda b$, $z = z_1+\lambda c$, giving the Cartesian symmetric form $(x-x_1)/a = (y-y_1)/b = (z-z_1)/c$ (NCERT §11.3.1, p. 382). Each fraction equals the common parameter λ.
- If l, m, n are DCs of the line, the same equation reads $(x-x_1)/l = (y-y_1)/m = (z-z_1)/n$ (NCERT §11.3.1 Note, p. 382). DC form is occasionally more natural when angles are given directly.
- The angle θ between two lines with DRs a_1, b_1, c_1 and a_2, b_2, c_2 satisfies $\cos\theta = (a_1a_2 + b_1b_2 + c_1c_2) / [\sqrt{a_1^2 + b_1^2 + c_1^2} \cdot \sqrt{a_2^2 + b_2^2 + c_2^2}]$ (NCERT §11.4 eqn (1), p. 383).
- In DC form (since $l^2 + m^2 + n^2 = 1$): $\cos\theta = |l_1l_2 + m_1m_2 + n_1n_2|$ (NCERT §11.4 eqn (3), p. 384). The absolute value enforces the acute-angle convention.
- Two lines are perpendicular iff $a_1a_2 + b_1b_2 + c_1c_2 = 0$; parallel iff $a_1/a_2 = b_1/b_2 = c_1/c_2$ (NCERT §11.4, p. 384). These are vector statements: perpendicular ⇔ dot product 0; parallel ⇔ direction vectors collinear.
- In vector form, for $r = a_1 + \lambda b_1$ and $r = a_2 + \mu b_2$, the acute angle satisfies $\cos\theta = |b_1 \cdot b_2| / (|b_1||b_2|)$ (NCERT §11.4, p. 384).
- Skew lines are non-coplanar lines in space — neither intersecting nor parallel; the shortest distance segment between them is perpendicular to both (NCERT §11.5, p. 385–386). The room-corner example (Fig 11.5) makes this geometrically intuitive.
- Shortest distance between skew lines $r = a_1 + \lambda b_1$ and $r = a_2 + \mu b_2$ is $d = |(b_1 \times b_2) \cdot (a_2 - a_1)| / |b_1 \times b_2|$ (NCERT §11.5.1, p. 386–387). The numerator is the volume of the parallelepiped spanned by a_2-a_1, b_1, b_2; the denominator is the area of the base.
- Cartesian form of the shortest distance uses the scalar triple product (a determinant with rows $x_2-x_1, y_2-y_1, z_2-z_1$; a_1, b_1, c_1; a_2, b_2, c_2) divided by $\sqrt{(b_1c_2-b_2c_1)^2 + (c_1a_2-c_2a_1)^2 + (a_1b_2-a_2b_1)^2}$ (NCERT §11.5.1, p. 387). Two skew lines intersect iff this shortest distance is zero — equivalent to the determinant vanishing (the coplanarity condition).
- Distance between two parallel lines $r = a_1 + \lambda b$ and $r = a_2 + \mu b$ is $d = |b \times (a_2 - a_1)| / |b|$ (NCERT §11.5.2, p. 387–388). The cross product picks up the component of $(a_2 - a_1)$ perpendicular to b.

- Two parallel lines are coplanar; the distance equals the length of the perpendicular from any point of one to the other (NCERT §11.5.2, p. 387).
- Planes are out of scope in the current reprint; equations of planes, angle between planes, distance from a point to a plane, and intersection of a line with a plane are deferred to advanced study.
- A practical guide to picking the right formula: if the problem gives angles → DCs; if it gives a position and direction → vector form $r = a + \lambda b$; if it gives two points → two-point form; if it asks the angle → $\cos\theta$ formula; if it asks distance and lines are parallel → parallel distance formula; if skew → skew distance formula.
- The vector approach is computationally lighter than the Cartesian: dot/cross products handle direction-vector arithmetic uniformly. Convert to vector form whenever a problem looks messy in Cartesian.
- A practical tip for shortest distance: compute $b_1 \times b_2$ first; if it equals 0, lines are parallel; else they are either skew or intersecting. They intersect iff the scalar triple product $(a_2 - a_1)\cdot(b_1 \times b_2) = 0$; otherwise they are skew with positive shortest distance.
- Standard 3D hint: changing coordinates (e.g. translating the origin to one of the line's points) often simplifies a problem; work with vectors based at the origin where possible.

2.2 Definitions to memorise

Term	Definition	Page
Direction cosines	$l = \cos\alpha$, $m = \cos\beta$, $n = \cos\gamma$	377
Direction ratios	Any 3 numbers proportional to DCs	378
Identity for DCs	$l^2 + m^2 + n^2 = 1$	378
Direction-reversal	DCs flip sign	378
DCs from DRs	$l = \pm a/\sqrt{(a^2+b^2+c^2)}$ etc.	379
DRs from two points	$(x_2 - x_1, y_2 - y_1, z_2 - z_1)$	379
Skew lines	Non-coplanar, neither intersect nor parallel	385
Parallel lines	Same DR direction	384
Perpendicular lines	$a_1a_2 + b_1b_2 + c_1c_2 = 0$	384
Angle between lines (DR form)	$\cos\theta = (\Sigma a_i a_j)$/(prod of magnitudes)	383
Angle between lines (DC form)	$\cos\theta =$	$\Sigma\, l_i l_j$
Angle (vector form)	$\cos\theta =$	$b_1 \cdot b_2$
Vector equation of line	$r = a + \lambda b$	382
Cartesian equation of line	$(x - x_1)/a = (y - y_1)/b = (z - z_1)/c$	382
Parametric equations	$x = x_1 + \lambda a$, $y = y_1 + \lambda b$, $z = z_1 + \lambda c$	382

Term	Definition	Page
Position vector	a for point A	381
Shortest distance (skew)		$(b_1 \times b_2)\cdot(a_2-a_1)$
Shortest distance (parallel)		$b \times (a_2-a_1)$
Coplanar lines	Lie in a common plane	385
Coplanarity test	$(a_2-a_1)\cdot(b_1\times b_2) = 0$	387
Scalar triple product	[u v w] = u·(v × w)	386
Parallelepiped volume		[u v w]
Direction vector b	Parallel to the line	381
Two-point line (vector form)	$r = a_1 + \lambda(a_2 - a_1)$	381
Foot of perpendicular	Closest point of one line to another	386

2.3 Diagrams / processes to remember

- **Fig 11.1 (p. 378):** a directed line L making angles α, β, γ with the coordinate axes — the picture that justifies l = cos α etc.
- **Fig 11.2 (p. 379):** the line PQ between P(x1, y1, z1) and Q(x2, y2, z2) with perpendiculars to the XY-plane; gives the formula cos γ = (z2 – z1)/PQ.
- **Fig 11.3 (p. 381–382):** point A with position vector a, line through A parallel to b, arbitrary point P with position vector r — the geometric picture of r = a + λb.
- **Fig 11.4 (p. 383):** two lines through the origin with DRs a1, b1, c1 and a2, b2, c2 — used to derive the cos θ formula.
- **Fig 11.5 (p. 385):** the room of dimensions 1 × 3 × 2 with the diagonal GE on the ceiling and DB along the wall — the canonical example of skew lines.
- **Fig 11.6 (p. 386):** skew lines l1, l2 with the perpendicular segment PQ and the foot of perpendicular construction used to derive d = |(b1 × b2)·(a2 – a1)| / |b1 × b2|.
- **Fig 11.7 (p. 387):** parallel lines l1, l2 with foot of perpendicular from T to l1, justifying d = |b × (a2 – a1)| / |b|.
- **Process — find DCs:** identify DRs from problem; compute $\sqrt{(a^2+b^2+c^2)}$; divide each DR by the magnitude.
- **Process — angle between two lines:** identify direction vectors b_1, b_2 (or DRs); compute dot product and magnitudes; apply cos θ = $|b_1\cdot b_2|/(|b_1||b_2|)$.
- **Process — line equation through two points:** compute $b = a_2 - a_1$; use $r = a_1 + \lambda b$ or Cartesian symmetric form.
- **Process — shortest distance (skew):** compute $b_1 \times b_2$; compute $a_2 - a_1$; dot the cross with the difference; divide by the magnitude of the cross.

2.4 Common confusions / NTA trap points

- Mixing up DCs and DRs: only DCs satisfy $l^2 + m^2 + n^2 = 1$. If a question gives "direction ratios 2, −1, −2", do NOT just plug them in — first normalise to (2/3, −1/3, −2/3).
- Forgetting that DCs come with a sign ambiguity (line vs directed line). Numerically equal sets like (1/√3, 1/√3, 1/√3) and (−1/√3, −1/√3, −1/√3) refer to the same line in opposite directions.
- Using cos θ = b1·b2 / (|b1||b2|) without absolute value — θ is taken to be the acute angle, so |b1·b2| is required.
- Confusing the shortest-distance formula for skew lines (scalar triple over |b1 × b2|) with the parallel-line formula (|b × (a2 − a1)|/|b|). Apply the parallel formula only after checking that b1, b2 are proportional.
- Reading the symmetric Cartesian form: the numerators give the point on the line (x1, y1, z1), the denominators give the DRs (a, b, c). Students often swap these.
- Treating $l^2 + m^2 + n^2 = 1$ as a "check" rather than a constraint: when DCs are asked, you must divide DRs by their magnitude; you cannot just report the DRs as DCs.
- Forgetting that two parallel direction vectors are proportional, not equal — the proportionality test $a_1/a_2 = b_1/b_2 = c_1/c_2$ is the correct one.
- Using the skew-line formula when lines are parallel — $b_1 \times b_2 = 0$, so the formula blows up. Always test for parallelism first.
- Sign-error in cross product: $b_1 \times b_2 = -(b_2 \times b_1)$. The formula for distance uses absolute value, so the sign does not change the answer, but it does matter for directed-distance calculations.
- Mis-reading the parametric form: $(x - x_1)/a = (y - y_1)/b = (z - z_1)/c$, NOT $(x - a)/x_1$. The constants on the side of x are the point coordinates; the denominators are direction components.
- Confusing parallel and coincident lines. Coincident means same line; parallel means non-intersecting with same direction.
- Treating the shortest-distance formula as a magnitude of a vector. It is a scalar.

2.5 Key formulas & theorems

Formula	Statement	NCERT page
DCs	$l = \cos\alpha$, $m = \cos\beta$, $n = \cos\gamma$	377
Identity	$l^2 + m^2 + n^2 = 1$	378
DRs to DCs	$l = \pm a/\sqrt{(a^2+b^2+c^2)}$	379
Two-point DRs	$(x_2-x_1, y_2-y_1, z_2-z_1)$	379
Two-point DCs	$(\Delta x)/PQ, (\Delta y)/PQ, (\Delta z)/PQ$	380

Formula	Statement	NCERT page
PQ length	$\sqrt{(\Sigma(\Delta)^2)}$	380
Vector eq. (point + dir)	$r = a + \lambda b$	382
Cartesian eq. (symmetric)	$(x-x_1)/a = (y-y_1)/b = (z-z_1)/c$	382
Vector eq. (2 points)	$r = a_1 + \lambda(a_2 - a_1)$	382
Cartesian (2 points)	$(x-x_1)/(x_2-x_1) = (y-y_1)/(y_2-y_1) = (z-z_1)/(z_2-z_1)$	382
Angle (DR)	$\cos\theta = (a_1a_2 + b_1b_2 + c_1c_2)/(\text{prod mag})$	383
Angle (DC)	$\cos\theta =$	$l_1l_2 + m_1m_2 + n_1n_2$
Angle (vector)	$\cos\theta =$	$b_1 \cdot b_2$
Perpendicular condition	$a_1a_2 + b_1b_2 + c_1c_2 = 0$	384
Parallel condition	$a_1/a_2 = b_1/b_2 = c_1/c_2$	384
Shortest distance (skew)		$(b_1 \times b_2)\cdot(a_2-a_1)$
Shortest distance (parallel)		$b \times (a_2-a_1)$
Coplanarity test	$(a_2-a_1)\cdot(b_1 \times b_2) = 0$	387
Skew lines	Non-coplanar	385
Cross product magnitude		$b_1 \times b_2$
Dot product	$a\cdot b =$	a
Scalar triple product	$[u\ v\ w] = u\cdot(v \times w)$	386
Line through origin	$r = \lambda b$	382
$\sin^2 + \cos^2$ between lines	1 (rotation invariant)	384
Volume parallelepiped		scalar triple product
Perpendicular from point to line	Standard projection formula	387

2.6 Solved examples (NCERT-grounded)

Example A (NCERT Example 1, p. 380). DCs of a line making angles 90°, 60°, 30° with axes.

Step 1 — write cosines: $l = \cos 90° = 0$; $m = \cos 60° = 1/2$; $n = \cos 30° = \sqrt{3}/2$. Step 2 — verify identity: $0 + 1/4 + 3/4 = 1$ ✓. Step 3 — answer: **$(0, 1/2, \sqrt{3}/2)$**.

Example B (NCERT Example 2, p. 380). DCs from DRs 2, −1, −2.

Step 1 — magnitude: $\sqrt{(4 + 1 + 4)} = 3$. Step 2 — divide: $(2/3, -1/3, -2/3)$. Step 3 — check identity: $4/9 + 1/9 + 4/9 = 1$ ✓. **Answer:** $(2/3, -1/3, -2/3)$.

Example C (NCERT Example 6, p. 382). Equation of line through (5, 2, –4) parallel to $3\hat{i} + 2\hat{j} - 8\hat{k}$.

Step 1 — vector form: $\vec{r} = 5\hat{i} + 2\hat{j} - 4\hat{k} + \lambda(3\hat{i} + 2\hat{j} - 8\hat{k})$. Step 2 — Cartesian form: $(x - 5)/3 = (y - 2)/2 = (z + 4)/-8$. Step 3 — state: both forms above.

Example D (NCERT Example 7, p. 384). Angle between $\vec{r} = 3\hat{i} + 2\hat{j} - 4\hat{k} + \lambda(\hat{i} + 2\hat{j} + 2\hat{k})$ and $\vec{r} = 5\hat{i} - 2\hat{j} + \mu(3\hat{i} + 2\hat{j} + 6\hat{k})$.

Step 1 — $b_1 \cdot b_2$: $1 \cdot 3 + 2 \cdot 2 + 2 \cdot 6 = 19$. Step 2 — magnitudes: $|b_1| = 3$, $|b_2| = 7$; product = 21. Step 3 — cosine: $\cos\theta = 19/21$; $\boldsymbol{\theta = \cos^{-1}(19/21)}$.

Example E (NCERT Example 9, p. 388). Shortest distance between $\vec{r} = \hat{i} + \hat{j} + \lambda(2\hat{i} - \hat{j} + \hat{k})$ and $\vec{r} = 2\hat{i} + \hat{j} - \hat{k} + \mu(3\hat{i} - 5\hat{j} + 2\hat{k})$.

Step 1 — cross product: $b_1 \times b_2 = 3\hat{i} - \hat{j} - 7\hat{k}$; magnitude = $\sqrt{(9 + 1 + 49)} = \sqrt{59}$. Step 2 — $a_2 - a_1$: $\hat{i} + 0\hat{j} - \hat{k}$; dot with cross = 3 + 0 + 7 = 10. Step 3 — divide: $d = 10/\sqrt{59}$. **Answer:** $10/\sqrt{59}$.

🎯 Practice MCQs

📊 PYQ Alignment

> CUET (UG) Mathematics has been pulling 2–3 items per year from this unit, with a strong bias toward direct application of the $\cos\theta$ formula for the angle between two lines (DR form) and the shortest-distance formula for skew lines. Expect at least one MCQ on converting DRs to DCs (or vice versa), one on writing the vector/Cartesian equation of a line through a given point with a given direction, and one numerical on shortest distance between skew or parallel lines — almost always reusing the structure of NCERT Examples 6, 7, 9 and 10.